Lecture 16

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Derivatives and shapes of graphs

What does f' say about f ?

Increasing / Decreasing Test

a) If $f'(x) > 0$ on an interval, then f is increasing on that interval

b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval

Pf a) Let x_1 and x_2 be any two numbers in the interval with $x_1 < x_2$

So to show that the function is increasing, we have to show that $f(x_1) < f(x_2)$

Since $f'(x) > 0$, we Know that f is differentiable on (x_1, x_2)

So by MVT, there is a number c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $x_1 < x_2$, $x_2 - x_1 > 0$ and by assumption $f'(c) > 0$, so the right hand side is positive.

Therefore, $f(x_2) - f(x_1) > 0$ as desired.

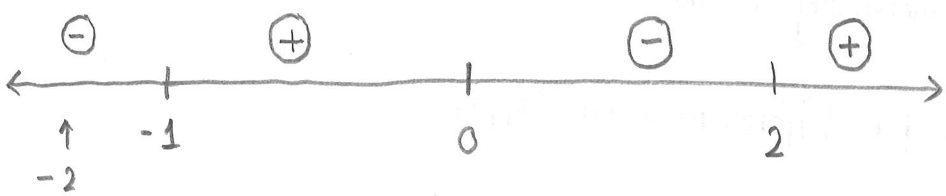
b) Can be proved similarly.

Example Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Soln $f'(x) = 12x^3 - 12x^2 - 24x = 12x\,(x^2 - x - 2)$

$= 12x\,(x-2)(x+1)$

So the critical numbers are when $f'(c) = 0$ or $f'(c)$ does not exist. In this case since $f'(x)$ is always defined, the critical numbers are $0, 2, -1$.



So we need to determine what happens in each interval

$f'(2) = 12(-2)(-2-2)(-2+1) < 0$

So function increasing on $(-1, 0) \cup (2, \infty)$

function decreasing on $(-\infty, -1) \cup (0, 2)$ ☒

Recall that if c is a local max / min then it is a critical number, but not every critical number is a local max/min. So we would like to tell when a critical number is a local max/min.

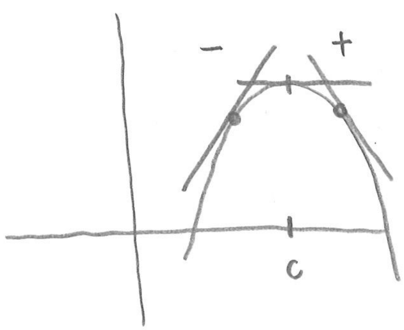
THE FIRST DERIVATIVE TEST

Spse that c is a critical number of a continuous function f

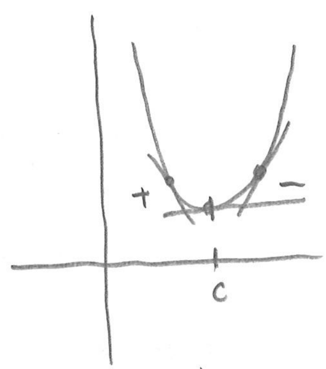
a) If f' changes from positive to negative at c, then f has a local max at c.

b) If f' changes from negative to positive at c, then f has a local min at c.

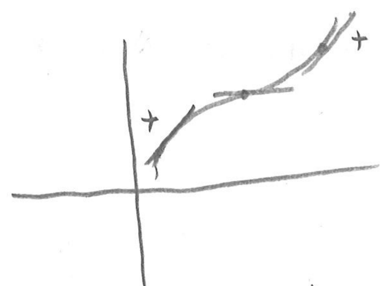
c) If f' does not change at c (i.e f' is positive on both sides of c, or negative on both sides of c), then f has no local max/min at c.



Local max



Local min



Neither local max nor min

Ex 1 local max at -1, $f(-1) = 0$

local min at 0, $f(0) = 5$

local max at 2, $f(2) = -27$

Ex 3 Find the local max and minimum values of the function

$$g(x) = x + 2\sin x \quad , \quad 0 \le x \le 2\pi$$

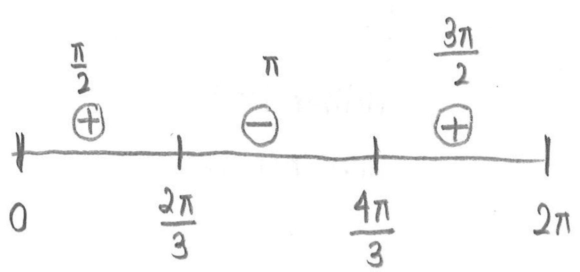
Sol^n $g'(x) = 1 + 2\cos x$

To find critical numbers set $g'(x) = 0$

$$1 + 2\cos x = 0 \Rightarrow \cos x = -\frac{1}{2}$$

Since we are only worried about interval $[0, 2\pi]$,

the solution of the equation is $x = \frac{2\pi}{3}, \frac{4\pi}{3}$



$$1 + 2\cos\left(\frac{\pi}{2}\right) = 1$$

$$1 + 2\cos(\pi) = -1$$

$$1 + 2\cos\left(\frac{3\pi}{2}\right) = 1$$

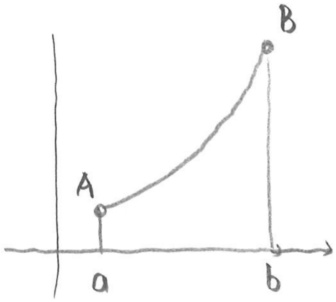
So increasing on $\left(0, \frac{2\pi}{3}\right) \cup \left(\frac{4\pi}{3}, 2\pi\right)$

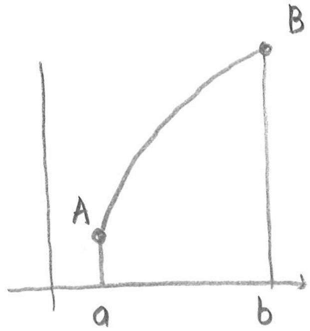
Decreasing on $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$

local min at $\frac{2\pi}{3}$, $g\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + 2\sin\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + 2 \cdot \frac{\sqrt{3}}{2} \approx 3.83$

local max at $\frac{4\pi}{3}$, $g\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + 2\sin\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} - 2 \cdot \frac{\sqrt{3}}{2} \approx 2.46$

What does f'' say about f ?

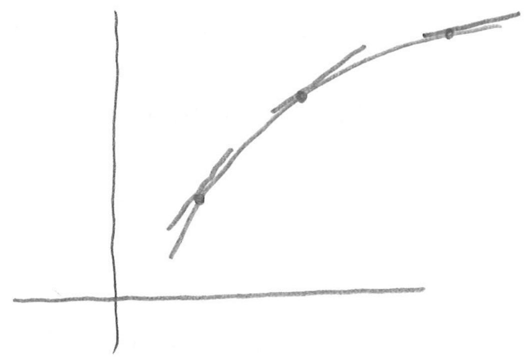




Both functions are increasing but they bend in different directions.

Concave upward :

The curve lies above the tangent lines



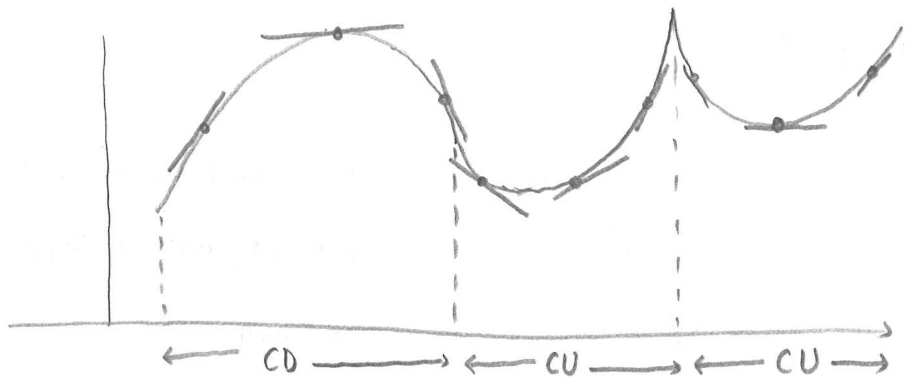
Concave downward :

The curve lies below the tangent lines

DEFINITION

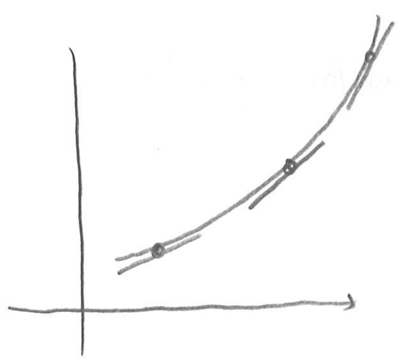
If the graph of f lies above all of it's tangents on an interval I, then it is called concave upward on I.

If the graph of f lies below all of it's tangent lines on an interval I, then it is called concave downward on I

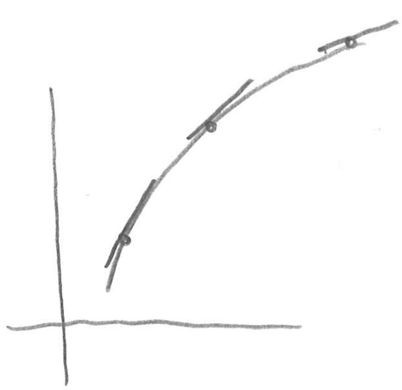


<u>Defn</u> A point P on a curve $y = f(x)$ is called an <u>inflection point</u> if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward.

So how does second derivative help us determine interval of concavity ??



Slope of tangent line is increasing
That means f' is increasing function,
which means that its derivative is
positive i.e $f'' > 0$.



Slope of tangent line is decreasing.
That means f' is a decreasing function
which means it's derivative is negative
i.e. $f'' < 0$.

Concavity Test

a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

So inflection point is where the second derivative changes sign.

Another
Application of Second derivative

The Second Derivative Test

Suppose f'' is continuous near c.

a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c

b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at c

(For ex, part a is true because $f''(x) > 0$ near c so f is concave upward near c. This means that the graph of f lies above its horizontal tangent at c and so f has a local min at c.)

Ex $f(x) = x^4 - 4x^3$

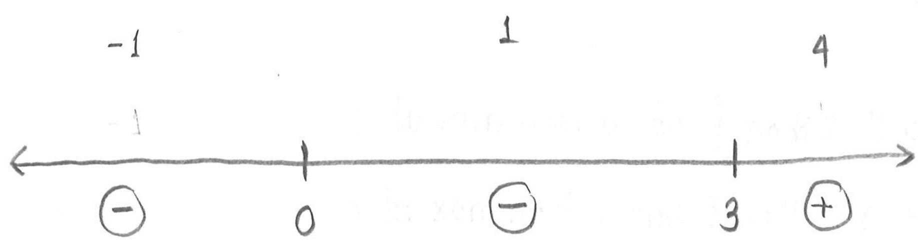
$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

$f''(x) = 12x^2 - 24x = 12x(x-2)$

Critical numbers

$f'(x) = 0 \Rightarrow 4x^2(x-3) = 0 \Rightarrow 4x^2 = 0$ or $x - 3 = 0 \Rightarrow x = 0, 3$

Intervals of Increase / Decrease



Increasing on $(3, \infty)$

Decreasing on $(-\infty, 0) \cup (0, 3)$

Max / Min

Local max at 3, $f(3) = -27$

Neither local min nor max at 0

First Derivative Test

Verify answers using Second Derivative Test

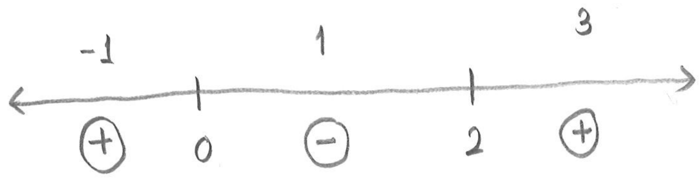
$f''(3) = 36 > 0$ and $f'(3) = 0$, so local min at $f(3) = -27$

$f''(0) = 0$ and $f'(0) = 0$, so no information obtained.

Concavity

Repeat the same process as for the case of I/D but with 2^{nd} derivative

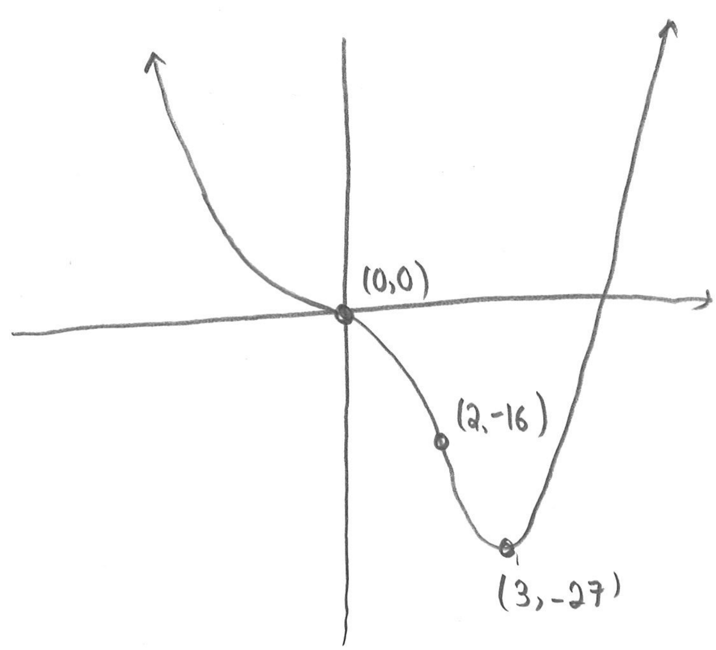
- $f''(x) = 0 \Rightarrow 12x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$



CU on $(-\infty, 0) \cup (2, \infty)$

CD on $(0, 2)$

Inflection point $(0,0)$, $(2,-16)$



Ex Sketch a graph of a function f that satisfies the following conditions :

a) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$

b) $f''(x) > 0$ on $(-\infty, -2) \cup (2, \infty)$, $f''(x) < 0$ on $(-2, 2)$

c) $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 0$